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# Mutual synchronization between quantum-dot passively mode-locked laser frequency combs: Theoretical approach

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## Abstract

A comprehensive theoretical framework is established for describing repetition-rate synchronization and partial optical coherence in two bidirectionally coupled quantum-dot passively mode-locked lasers (QD-PMLs). Each laser is modeled as a two-section monolithic device composed of a forward-biased gain section and a reverse-biased saturable absorber. The coupled delay-rate equations incorporate the slowly varying complex optical fields, modal gain dynamics, absorber loss, photon decay, linewidth-enhancement factors, delayed optical coupling, coupling phase, and spontaneous-emission noise. In the weak mutual-injection limit, the full model is systematically reduced to an effective timing-phase description, yielding the locking relation  $\Delta\Omega_0 = 2K\cos(\psi - \Omega_s\tau_{\text{ext}})\sin(\delta)$ , and hence the locking condition  $|\Delta\Omega_0| \leq 2K|\cos(\Omega_s\tau_{\text{ext}} - \psi)|$ . Representative simulations establish that two initially detuned 10 GHz pulse trains are pulled into a single shared repetition frequency, with the locked frequency showing pronounced and controllable delay-dependent tunability. The synchronized state exhibits a marked reduction in root-mean-square timing jitter, whereas finite mutual optical coherence appears only when the optical carrier mismatch simultaneously lies within the coherent injection-locking bandwidth. Agreement with recent synchronization experiments on InAs/InGaAs QD-PMLL combs demonstrates that the model not only reproduces the central experimental trends but also clarifies the underlying hierarchy: repetition-rate locking is robust and readily achieved, delay tuning provides an effective control parameter, timing noise is strongly suppressed in the locked regime, and optical phase coherence is an additional conditional constraint rather than an automatic consequence of rate synchronization.

**Keywords:** quantum-dot laser, passively mode-locked laser, frequency comb, delayed optical coupling, synchronization, timing jitter, mutual coherence.

## 1. Introduction

Quantum-dot passively mode-locked lasers (QD-PMLs) represent a highly attractive class of compact semiconductor sources for generating picosecond pulse trains and optical frequency combs. Owing to their low-

dimensional active medium, these devices exhibit discrete carrier states, reduced temperature sensitivity, enhanced differential gain, and rich dynamical behavior that is fundamentally distinct from that of conventional quantum-well lasers. These advantages position QD-PMLLs as promising light sources for integrated photonic interconnects, microwave photonics, coherent optical communication systems, high-speed optical sampling, and chip-scale metrology [1]. Early demonstrations of stable mode locking in two-section QD lasers, together with extensive studies of mode-locked QD and quantum-dash semiconductor lasers, have firmly established low-dimensional gain media as a robust platform for compact comb generation [2–4]. Furthermore, nonlinear dynamical analyses have revealed that carrier reservoirs, carrier capture and escape processes, and the associated time scales critically govern pulse formation, timing stability, and the overall mode-locking dynamics of these devices [5].

Recent experimental demonstrations further support the use of QD active media as practical comb sources for integrated photonics. Two-section InAs QD mode-locked lasers have produced combs with a stable 25.5 GHz spacing over a wide temperature range, while broadband QD devices have been shown to operate in both amplitude-modulated and frequency-modulated comb regimes [6,7]. The direct growth of reflection-insensitive QD lasers on silicon, together with continuing advances in silicon photonics, also strengthens their relevance for scalable photonic integration [8,9]. More recently, high-channel-count passively mode-locked QD lasers monolithically integrated on silicon have enabled multi-terabit transmission, highlighting their potential for integrated communication systems [10]. These advances show that QD comb sources can function not only as short-pulse emitters, but also as stable, broadband, multi-line sources for chip-scale optical platforms.

Synchronization is a critical requirement in systems where multiple optical combs operate together. Optical injection locking offers an all-optical mechanism for aligning the frequency and phase dynamics of semiconductor lasers, avoiding the need for high-power microwave distribution [11]. At the same time, studies of optically injected semiconductor lasers have shown that injection strength, frequency detuning, and relative phase can substantially reshape the nonlinear laser dynamics, including transitions to complex or chaotic behavior [12]. Earlier work on optical-pulse injection locking, together with general frequency-comb injection-locking theory, further indicates that repetition-rate locking and optical carrier-phase compatibility are closely related but not identical synchronization conditions [13–15]. For QD-PMLLs, optical feedback and delayed optical perturbations are particularly important because they can directly influence phase noise, RF linewidth, and pulse-to-pulse timing jitter [16,17]. From a modeling perspective, delayed-field formulations are rooted in the Lang–Kobayashi description of external optical feedback in semiconductor lasers [18], providing the basis for incorporating delayed optical coupling terms into the present rate-equation framework.

In many practical systems, a single optical comb source is not sufficient; multiple combs must operate with matched line spacing, stable relative pulse timing, or controlled mutual coherence. Electrical locking can achieve this synchronization, but it requires microwave driving circuitry and may reduce the scalability and compactness of integrated photonic implementations. Optical synchronization provides a more attractive alternative by allowing the pulse train or comb field emitted by one laser to be injected into another, thereby imposing timing and/or frequency coherence through purely optical coupling. Recent experiments have demonstrated synchronization of QD mode-locked laser combs under unidirectional optical injection [19], followed by bidirectional passive optical coupling between two monolithic InAs/InGaAs QD-PMLLs operating near 10 GHz and 1260 nm [20]. These demonstrations motivate a dynamical modeling framework capable of capturing how

delayed optical coupling, coupling strength, detuning, and mutual phase interactions govern synchronization between QD-PMLLs.

These experimental advances raise a theoretical question of direct relevance to both device design and system integration: what is the minimal physically grounded model that links delayed optical coupling to repetition-rate locking, delay-dependent frequency tuning, timing-jitter suppression, and conditional optical coherence? The approach adopted here is deliberately reduced, yet remains physically traceable. Specifically, this manuscript extends standard delay-rate descriptions of passive mode locking [21,22] by incorporating bidirectional delayed optical injection together with a timing-phase reduction. This formulation is consistent with reduced synchronization frameworks previously used to describe coupled photonic oscillators and micro resonator combs [23].

The objective of this work is to develop a theoretical framework that directly links coupled QD-PMLL rate equations to the synchronization signatures observed experimentally. The model is maintained in symbolic form so that each variable and parameter can be clearly associated with a measurable physical quantity or a design-relevant device parameter. In this form, the framework serves as a compact analytical design tool for estimating synchronization and locking conditions prior to fully calibrated numerical simulations or experimental optimization.

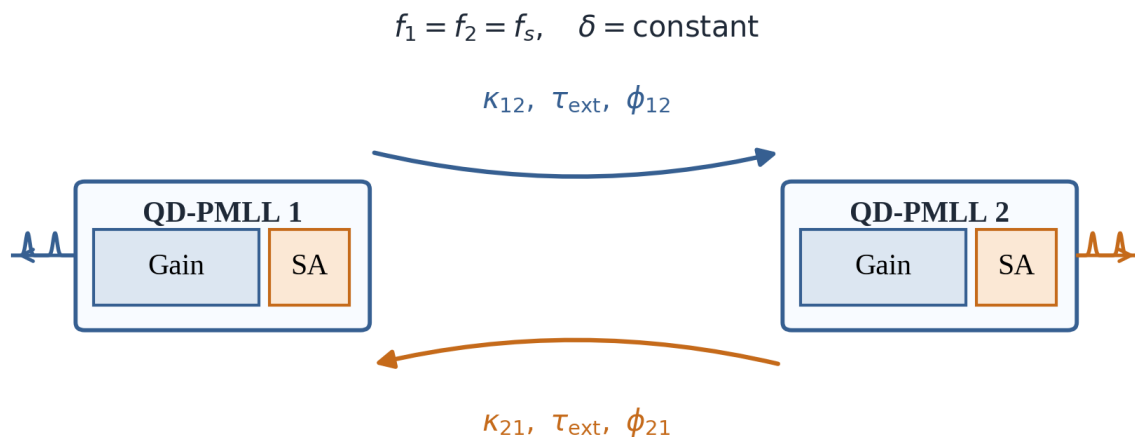


Figure 1. Bidirectionally coupled QD-PMLL frequency-comb system. The two lasers exchange delayed optical fields with coupling coefficients  $\kappa_{12}$  and  $\kappa_{21}$ , external delay  $\tau_{\text{ext}}$ , and phases  $\phi_{12}$  and  $\phi_{21}$ . In the locked state, the repetition frequencies collapse to  $f_1 = f_2 = f_s$  with a constant timing offset  $\delta$ .

## Nomenclature

Subscripts  $i$  and  $j$  stands for the receiving and the injecting lasers, respectively.

Table 1. Nomenclature used in the model.

| Symbol                                       | Definition                                                        | Unit                      |
|----------------------------------------------|-------------------------------------------------------------------|---------------------------|
| $i, j$                                       | laser indices; $j$ denotes the opposite laser to $i$              | dimensionless             |
| $t$                                          | time                                                              | s                         |
| $E_i(t)$                                     | slowly varying complex intracavity field of laser $i$             | normalized field          |
| $P_i(t) =  E_i(t) ^2$                        | intracavity optical intensity or power                            | normalized power          |
| $g_i(t)$                                     | saturable modal gain                                              | $s^{-1}$ or dimensionless |
| $q_i(t)$                                     | saturable absorber loss                                           | $s^{-1}$ or dimensionless |
| $g_{0i}, q_{0i}$                             | unsaturated gain and absorber loss                                | $s^{-1}$ or dimensionless |
| $I_i, V_i$                                   | gain-section current and absorber reverse bias                    | A, V                      |
| $\tau_g, \tau_q$                             | gain and absorber recovery times                                  | s                         |
| $E_g, E_q$                                   | gain and absorber saturation energies                             | J or normalized           |
| $\gamma_p$                                   | photon decay rate                                                 | $s^{-1}$                  |
| $\Gamma$                                     | field-filter or cavity bandwidth parameter                        | $s^{-1}$                  |
| $R_i$                                        | linear round-trip transmission in the delay-map form              | dimensionless             |
| $\alpha_g, \alpha_q$                         | linewidth-enhancement factors of gain and absorber                | dimensionless             |
| $T_i, f_{r,i0}$                              | cavity round-trip time and free-running repetition frequency      | s, Hz                     |
| $\Omega_{i0} = 2\pi f_{r,i0}$                | free-running angular repetition frequency                         | rad $s^{-1}$              |
| $\Delta f_0 = f_{r,20} - f_{r,10}$           | free-running repetition-rate detuning                             | Hz                        |
| $\Delta\Omega_0 = \Omega_{20} - \Omega_{10}$ | free-running angular detuning                                     | rad $s^{-1}$              |
| $\kappa_{ij}$                                | optical coupling coefficient from laser $j$ into laser $i$        | $s^{-1}$ or normalized    |
| $\tau_{ij}$                                  | external propagation delay from laser $j$ into laser $i$          | s                         |
| $\tau_{\text{ext}}$                          | symmetric external propagation delay                              | s                         |
| $\phi_{ij}$                                  | optical phase accumulated along the coupling path                 | rad                       |
| $\xi_i(t), D_i$                              | complex spontaneous-emission noise and strength                   | model dependent           |
| $\theta_i(t)$                                | timing phase of the pulse train emitted by laser $i$              | rad                       |
| $\eta_i(t)$                                  | timing noise in the reduced phase equation                        | rad $s^{-1}$              |
| $K_{ij}, \psi_{ij}$                          | effective timing-coupling coefficient and phase shift             | rad $s^{-1}$ , rad        |
| $K_r$                                        | linear timing restoring rate in the locked state                  | $s^{-1}$                  |
| $\Omega_s, f_s$                              | synchronized angular and ordinary repetition frequencies          | rad $s^{-1}$ , Hz         |
| $\delta$                                     | constant timing-phase offset in the locked state                  | rad                       |
| $\vartheta_r$                                | small relative timing-phase perturbation about the locked state   | rad                       |
| $t_{i,n}, T_{i,n}$                           | arrival time of pulse $n$ and instantaneous period                | s                         |
| $\sigma_{T,i}$                               | root-mean-square timing jitter of laser $i$                       | s                         |
| $\gamma_{12}(\tau_c)$                        | normalized mutual optical coherence at correlation delay $\tau_c$ | dimensionless             |

| Symbol        | Definition             | Unit |
|---------------|------------------------|------|
| $S_i(\omega)$ | optical power spectrum | a.u. |

## 2. Coupled delay-rate model

### 2.1. Device configuration

The modeled system consists of two edge-emitting QD-PMLLs labelled  $i = 1$  and  $i = 2$ . The optical field inside laser  $i$  is denoted by  $E_i(t)$ , the corresponding intensity is  $P_i(t) = |E_i(t)|^2$ , the modal gain is  $g_i(t)$ , and the saturable absorber loss is  $q_i(t)$ . The gain section is governed by the injection current  $I_i$  through the unsaturated gain  $g_{0i}(I_i)$ , while the absorber section is controlled by the reverse-bias voltage  $V_i$  through the unsaturated loss  $q_{0i}(V_i)$ .

The optical field emitted by laser  $j$  is injected into laser  $i$  after an external delay  $\tau_{ij}$ , with optical coupling coefficient  $\kappa_{ij}$  and accumulated phase  $\phi_{ij}$ . Symmetric coupling is recovered under the conditions  $\kappa_{12} = \kappa_{21} = \kappa$ ,  $\tau_{12} = \tau_{21} = \tau_{\text{ext}}$ , and  $\phi_{12} = \phi_{21} = \phi$ . The asymmetric coupling configuration is retained in the general formulation to capture the transition from mutual synchronization to unidirectional injection locking.

### 2.2. Dynamics of Local field, gain, and absorber

For  $i, j$  in  $\{1, 2\}$  and  $j \neq i$ , the local coupled rate equation for the complex field is

$$\frac{dE_i(t)}{dt} = \frac{1}{2} \left[ (1 + i\alpha_g)g_i(t) - (1 + i\alpha_q)q_i(t) - \gamma_p \right] E_i(t) + \kappa_{ij} \exp(i\phi_{ij}) E_j(t - \tau_{ij}) + \xi_i(t). \quad (1)$$

A minimal gain-absorber representation is

$$\frac{dg_i(t)}{dt} = \frac{g_{0i}(I_i) - g_i(t)}{\tau_g} - \frac{g_i(t)P_i(t)}{E_g}, \quad (2)$$

$$\frac{dq_i(t)}{dt} = \frac{q_{0i}(V_i) - q_i(t)}{\tau_q} - \frac{q_i(t)P_i(t)}{E_q}. \quad (3)$$

Equations (1) – (3) are formulated in a deliberately compact form to emphasize the essential dynamical mechanisms. If required, this reduced description can be extended by replacing the scalar gain and absorber equations with more comprehensive quantum-dot carrier-reservoir rate equations that explicitly account for the wetting layer, excited states, and ground states [24–26]. The delayed coupling contribution is kept at the complex-field level, since repetition-rate locking and optical coherence are intrinsically determined by the simultaneous evolution of the optical amplitude and phase.

### 2.3. Delay-map interpretation

For strongly pulsed operation, the complex optical field can be formulated in a delay-map representation over a single cavity round trip

$$E_i(t + T_i) = \sqrt{R_i} \exp \left\{ \frac{1}{2} \left[ (1 + i\alpha_g)G_i - (1 + i\alpha_q)Q_i \right] \right\} E_i(t) + C_{ij} E_j(t - \tau_{ij}) \exp(i\phi_{ij}). \quad (4)$$

This representation highlights the role of the injected pulse. A delayed pulse emitted by laser  $j$  perturbs the pulse-formation window of laser  $i$  and consequently shifts the arrival time of the subsequent pulse. Although this timing

shift is small under weak coupling, it accumulates over many round trips and can lead to repetition-rate locking when the coupling-induced correction compensates the free-running detuning.

### 3. Reduction of timing-phase

Let  $\theta_i(t)$  denote or denotes the slowly varying timing phase of the pulse train emitted by laser  $i$ . The timing phase advances by  $2\pi$  each time the laser emits one pulse. Under weak coupling and stable mode locking, projecting the delayed optical perturbation onto the neutral timing mode gives

$$\frac{d\theta_i}{dt} = \Omega_{i0} + K_{ij}\sin[\theta_j(t - \tau_{ij}) - \theta_i(t) + \psi_{ij}] + \eta_i(t), \quad j \neq i. \quad (5)$$

Here  $\Omega_{i0} = 2\pi f_{r,i0}$  is the free-running angular repetition frequency,  $K_{ij}$  is the effective timing-coupling coefficient,  $\psi_{ij}$  is a phase shift determined by the pulse shape and coupling phase, and  $\eta_i(t)$  is timing noise. For symmetric coupling,

$$K_{12} = K_{21} = K, \quad \tau_{12} = \tau_{21} = \tau_{\text{ext}}, \quad \psi_{12} = \psi_{21} = \psi. \quad (6)$$

In the locked state both lasers share the synchronized angular repetition frequency  $\Omega_s$ , and their timing phases are

$$\theta_1(t) = \Omega_s t, \quad \theta_2(t) = \Omega_s t + \delta. \quad (7)$$

Substitution of Eq. (7) into the symmetric timing equations gives the locking relation

$$\Delta\Omega_0 = 2K\cos(\psi - \Omega_s\tau_{\text{ext}})\sin(\delta), \quad (8)$$

and hence the repetition-rate locking condition

$$|\Delta\Omega_0| \leq 2K|\cos(\Omega_s\tau_{\text{ext}} - \psi)|. \quad (9)$$

The synchronized frequency is shifted by the coupling and delay phase. A useful approximate expression is

$$\Omega_s \approx \frac{\Omega_{10} + \Omega_{20}}{2} + K\sin(\psi - \Omega_s\tau_{\text{ext}})\cos(\delta), \quad f_s = \frac{\Omega_s}{2\pi}. \quad (10)$$

Equations (8) – (10) demonstrate that the locking range is not determined solely by the coupling magnitude. Instead, the external delay and optical phase can either enhance the effective timing-restoring strength or weaken it.

### 4. Numerical implementation

The coupled delay-rate equations were numerically integrated using stored histories of the delayed fields  $E_j(t - \tau_{ij})$ . The integration time step was selected to resolve the shortest relevant physical time scale, including the absorber recovery time, the internal pulse width, and the delayed feedback contribution. Each simulation was separated into a transient interval and a recording interval. During the recording interval, the complex optical fields, intensities, gain, absorber loss, and pulse-arrival times were recorded.

Representative baseline parameters were selected on the experimental scale of QD-PMLLs operating at a repetition frequency of approximately 10 GHz and a center wavelength close to 1260 nm. These parameters are used as illustrative values and are not intended to represent a device-specific parameter extraction.

Table 2. Representative model parameters for the baseline theoretical simulations.

| Parameter                                    | Symbol              | Representative value   |
|----------------------------------------------|---------------------|------------------------|
| Center wavelength                            | $\lambda_0$         | 1260 nm                |
| Free-running repetition frequency of laser 1 | $f_{r,10}$          | 9.998 GHz              |
| Free-running repetition frequency of laser 2 | $f_{r,20}$          | 10.012 GHz             |
| Free-running detuning                        | $\Delta f_0$        | 14 MHz                 |
| Gain recovery time                           | $\tau_g$            | 800 ps                 |
| Absorber recovery time                       | $\tau_q$            | 5 ps                   |
| Gain linewidth-enhancement factor            | $\alpha_g$          | representative         |
| Absorber linewidth-enhancement factor        | $\alpha_q$          | representative         |
| External coupling delay                      | $\tau_{\text{ext}}$ | 12.5 ns                |
| Symmetric optical coupling coefficient       | $\kappa$            | varied                 |
| Symmetric timing-coupling coefficient        | $K$                 | fitted from simulation |
| Noise strength                               | $D_i$               | normalized             |

#### 4.1. Pulse-arrival time and locking

The  $n$ -th pulse-arrival time of laser  $i$  is denoted by  $t_{i,n}$  and is obtained from the local maximum of  $P_i(t)$  near that pulse. The instantaneous period is

$$T_{i,n} = t_{i,n+1} - t_{i,n}. \quad (11)$$

The average repetition frequency is

$$f_i = \left\langle \frac{1}{T_{i,n}} \right\rangle_n. \quad (12)$$

The two lasers are considered repetition-rate locked when

$$|f_1 - f_2| < \Delta f_{\text{tol}}, \quad (13)$$

where  $\Delta f_{\text{tol}}$  is a selected tolerance. The mutual timing error is obtained by comparing corresponding delayed pulses and removing the nearest integer multiple of the period.

#### 4.2. Timing jitter and mutual coherence

The root-mean-square timing jitter of laser  $i$  is calculated from the fluctuations of  $T_{i,n}$  around its mean:

$$\sigma_{T,i} = \sqrt{\left\langle \left[ T_{i,n} - \langle T_{i,n} \rangle_n \right]^2 \right\rangle_n}. \quad (14)$$

The normalized mutual optical coherence is calculated from the complex fields as

$$\gamma_{12}(\tau_c) = \frac{\langle E_1^*(t) E_2(t + \tau_c) \rangle}{\sqrt{\langle |E_1(t)|^2 \rangle \langle |E_2(t + \tau_c)|^2 \rangle}}. \quad (15)$$

Repetition-rate locking requires bounded pulse-envelope timing. High optical coherence additionally requires carrier-frequency compatibility and a favorable optical coupling phase.

## 5. Theoretical results

### 5.1. Unlocked reference case

For the baseline free-running case, the detuning is  $\Delta f_0 = 14$  MHz. The relative timing phase  $\theta_2 - \theta_1$  therefore drifts approximately linearly, the timing error remains unbounded, and the radio-frequency spectra show two distinct peaks. The optical spectra remain comb-like but exhibit different line spacings.

### 5.2. Repetition-rate locking map

Increasing the coupling coefficient strengthens the delayed mutual timing correction between the two pulse trains. When  $K$  is below the detuning term in Eq. (9), the injected pulse train only perturbs the local pulse-formation dynamics and is insufficient to arrest the relative timing drift. Once  $K$  exceeds the effective detuning threshold, the accumulated drift is suppressed and the two pulse trains converge to a common synchronized repetition frequency.

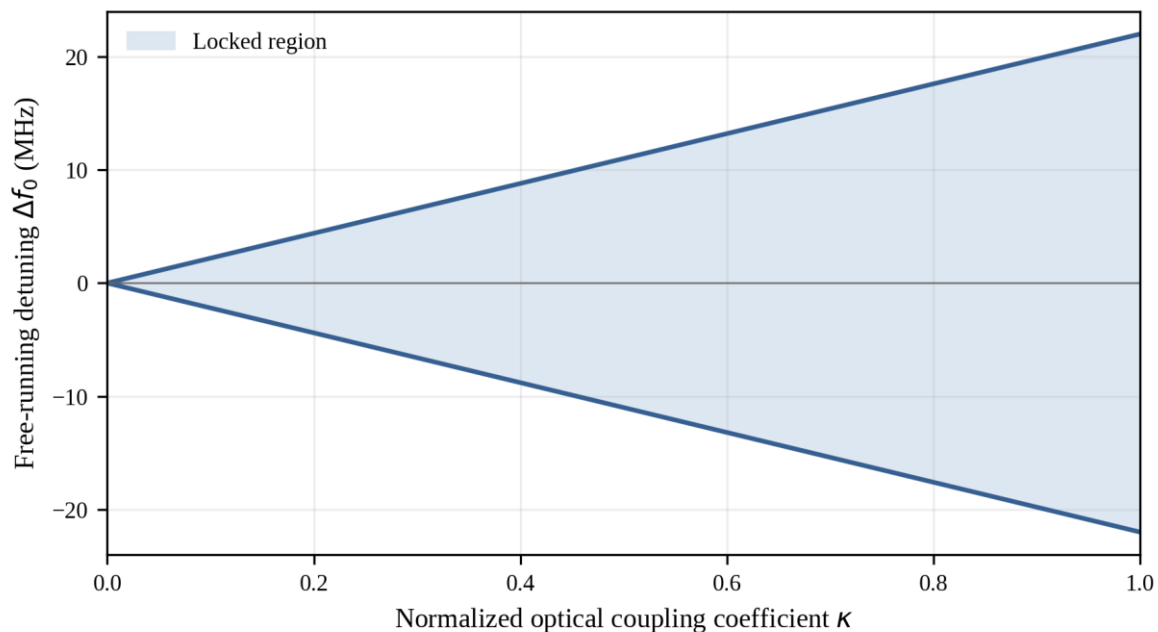


Figure 2. Theoretical synchronization map in the plane of normalized optical coupling coefficient  $\kappa$  and free-running detuning  $\Delta f_0$ . The wedge-shaped locked region reflects the synchronization boundary predicted by the reduced timing-phase condition in Eq. (9).

The predicted locking region has a wedge-like shape: small  $\kappa$  locks only small  $|\Delta f_0|$ , while larger  $\kappa$  increases the locking half-width. The shape is modified by the delay phase, so the same coupling magnitude can be effective or ineffective depending on  $\tau_{\text{ext}}$  and  $\phi$ .

### 5.3. Delay tuning of the common repetition frequency

Eq. (10) predicts that  $\tau_{\text{ext}}$  tunes  $f_s$  through the phase  $\Omega_s \tau_{\text{ext}} - \psi$ . From a physical perspective, a delayed pulse arriving ahead of the local pulse tends to advance its timing, whereas a pulse arriving after the local pulse tends to retard it. This behavior enables an all-optical mechanism for fine-tuning the synchronized microwave repetition frequency.

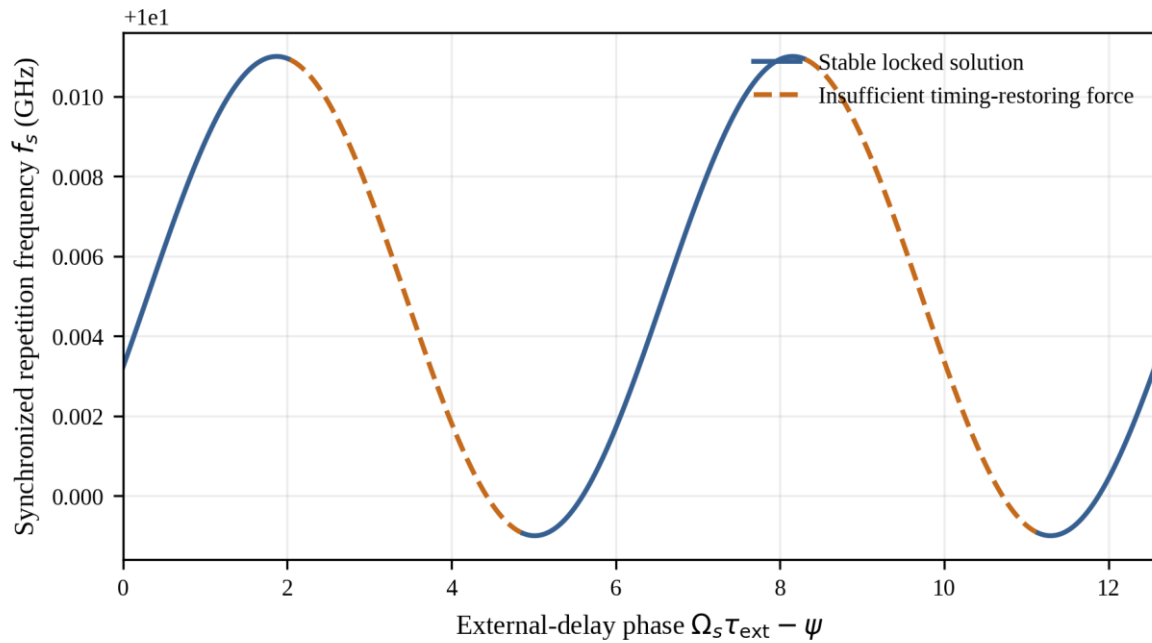


Figure 3. Representative delay-based tuning of the synchronized repetition frequency. The solid branch corresponds to a stable locked solution, while the dashed segments denote delay offsets for which the timing-restoring force is insufficient or has the incorrect sign.

#### 5.4. Timing-jitter reduction

In the unlocked state, spontaneous emission and carrier-induced amplitude-phase coupling produce timing diffusion. In the locked state, the relative timing perturbation  $\vartheta_r$  obeys the linearized restoring equation

$$\frac{d\vartheta_r}{dt} = -K_r \vartheta_r + \eta_r(t), \quad K_r > 0. \quad (16)$$

The relative timing-phase variance then scales as

$$\text{Var}(\vartheta_r) \approx \frac{D_\vartheta}{K_r}. \quad (17)$$

Thus, stronger positive restoring strength  $K_r$  reduces timing fluctuations, while unfavorable delay phases with  $K_r \leq 0$  cannot stabilize the locked state.

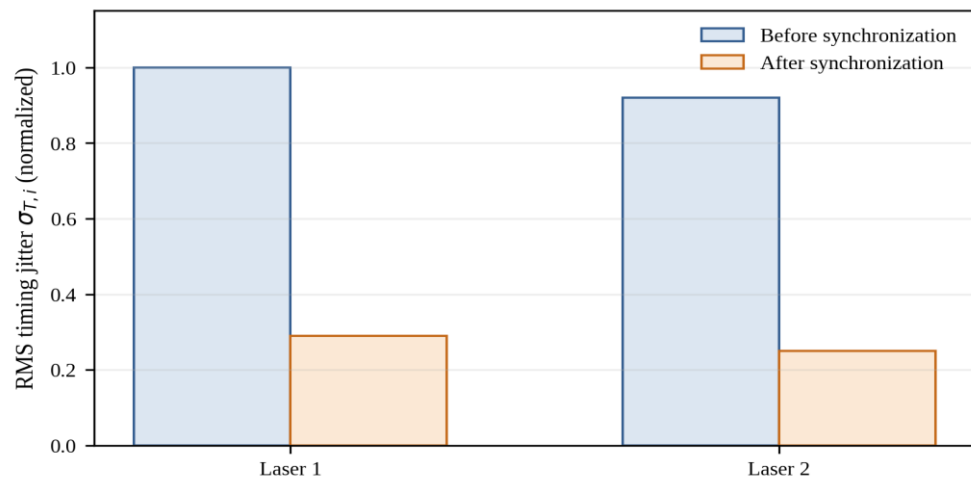


Figure 4. Representative RMS timing jitter  $\sigma_{T,i}$  before and after synchronization. These values are intended only as illustrative outputs of the normalized-noise model and must be recalibrated against experimentally measured timing-noise spectra before they can support reliable device-level predictions.

### 5.5. Mutual optical coherence

Repetition-rate locking constrains the pulse-envelope timing, but it does not by itself enforce stabilization of the optical carrier phase. The condition  $|\gamma_{12}|$  approximately equal to 1 requires the optical carrier-frequency mismatch to lie within the coherent injection-locking bandwidth and requires constructive coupling phase. Therefore, the model predicts a hierarchy: repetition-rate locking can occur without strong optical mutual coherence, approximately equal to 1 requires the optical carrier-frequency mismatch to remain within the coherent injection-locking bandwidth and further requires a constructive coupling phase. Accordingly, the model predicts a well-defined hierarchy: repetition-rate locking may be achieved even in the absence of strong optical mutual coherence, whereas high  $|\gamma_{12}|$  implies both rate locking and carrier-phase compatibility.

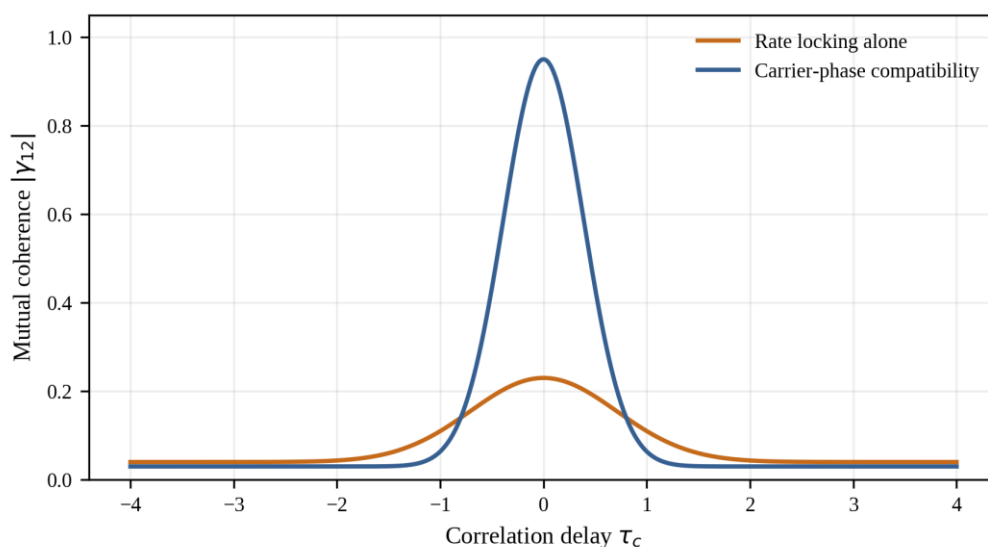


Figure 5. Calculated mutual coherence  $|\gamma_{12}|$  for two locked regimes. Rate locking alone gives a small coherence peak, while simultaneous carrier-phase compatibility gives a much stronger peak around the correlation delay  $\tau_c$ .

## 6. Comparison with experimental results

The model is evaluated against experimental trends reported for frequency-comb synchronization in QD mode-locked lasers. Both unidirectional synchronization with phase-noise reduction and bidirectional passive synchronization between monolithic InAs/InGaAs QD-PMLLs exhibit the same qualitative hierarchy: repetition-rate locking is robust, delay tuning is experimentally observable, locked states reduce relative timing jitter, and optical phase coherence emerges only when additional carrier-phase conditions are satisfied. The present theory is not intended to serve as a parameter-refitted reproduction of these specific devices; rather, it provides a compact framework that captures the same hierarchy using representative parameters on the relevant experimental scale.

Table 3. Comparison of the theoretical model with QD-PMLL synchronization experiments.

| Aspect            | Published experiment                                                           | Present model                                                                                             | Interpretation                                      |
|-------------------|--------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|-----------------------------------------------------|
| Device scale      | Two-section QD-PMLLs; approximately 10 GHz repetition rate; combs near 1260 nm | Baseline uses $f_{r,10} = 9.998$ GHz, $f_{r,20} = 10.012$ GHz, $\lambda_0 = 1260$ nm                      | Consistent operating scale                          |
| Coupling topology | Bidirectional passive coupling through a delayed optical path                  | $\kappa_{12}$ , $\kappa_{21}$ , $\tau_{12}$ , $\tau_{21}$ , $\phi_{12}$ , $\phi_{21}$ included explicitly | Same physical mechanism                             |
| Rate locking      | Different free-running rates collapse to identical repetition rates            | Eq. (9) predicts locking below the delay-weighted coupling bandwidth                                      | Agreement in mechanism                              |
| Delay tuning      | Common repetition rate can be tuned by external path length                    | Eq. (10) and Fig. 3 show $f_s$ depends on $\tau_{\text{ext}}$                                             | Agreement in trend                                  |
| Timing jitter     | Locked lasers exhibit reduced timing jitter relative to free-running operation | Linearized restoring force gives $\text{Var}(\vartheta_r) \approx D_\vartheta/K_r$                        | Agreement in sign and mechanism                     |
| Mutual coherence  | Interferometric fringes appear only under specific conditions                  | High $ \gamma_{12} $ only with carrier-frequency and phase compatibility                                  | Rate locking does not imply automatic phase locking |

### 6.1. Experimental extensions suggested by the model

First, a systematic scan of the external delay  $\tau_{\text{ext}}$  at fixed optical coupling would allow direct testing of the predicted delay-periodic locking bandwidth and delay-tuned synchronized repetition frequency.

Second, simultaneous measurement of RF linewidth, pulse-arrival-time jitter, and optical mutual coherence would separate envelope synchronization from carrier-phase coherence.

Third, controlled asymmetry between  $\kappa_{12}$  and  $\kappa_{21}$  would test the predicted transition from mutual synchronization to master-slave injection locking.

## 7. Physical interpretation

Bidirectional optical coupling synchronizes two QD-PMLLs by enforcing mutual pulse-timing correction. A pulse emitted from one laser reaches the other after the external delay, where it perturbs the optical field, gain, and absorber loss within the subsequent pulse-formation window. If these timing corrections are insufficient, the relative timing phase continues to drift because the free-running repetition frequencies remain mismatched. Once the coupling-induced corrections exceed this mismatch, the relative timing phase is driven toward a stable constant offset  $\delta$  and the two lasers emit pulse trains locked to a common repetition frequency.

The coupling delay serves two distinct functions. First, it sets which earlier pulse from one laser overlaps with the current pulse-formation window of the other laser. Second, it determines the phase factor governing the effective sign and magnitude of the timing correction. Consequently, even small variations in  $\tau_{\text{ext}}$  can shift the synchronization boundary and tune  $f_s$ .

Timing-jitter reduction occurs because mutual coupling converts the two lasers from independently diffusing timing oscillators into a jointly stabilized locked system. In the unlocked case, timing fluctuations in each QD-PMLL accumulate independently, leading to relative timing diffusion. In the locked state, however, the delayed optical field provides a restoring action: if a pulse is generated too early relative to the locked timing position, the incoming delayed field from the other laser tends to pull it back; if a pulse is generated too late, the same coupling mechanism tends to advance it. As a result, relative timing fluctuations are suppressed and the root-mean-square timing jitter is reduced. Optical coherence requires an even stricter condition than repetition-rate synchronization, because matching the pulse-envelope timing is not sufficient; the optical carrier phase must also remain stable within the coherent injection-locking bandwidth.

## 8. Limitations and future work

The reduced model provides a useful framework for interpreting the underlying synchronization mechanisms; however, quantitative prediction for a specific chip requires calibrated gain, absorber, noise, and coupling parameters. Future work should integrate the timing-phase model with measured device transfer functions, optical spectra, RF spectra, and interferometric coherence data. Further refinement may also include separate ground-state and excited-state carrier populations, wetting-layer capture dynamics, polarization effects, thermal drift, and spatial hole burning.

## 9. Conclusion

A coupled delay-rate and timing-phase theory is developed to describe mutual synchronization between quantum-dot passively mode-locked laser frequency combs. The complex-field formulation captures delayed bidirectional optical coupling between two two-section QD-PMLLs, including the key dynamical contributions of the optical fields, gain recovery, absorber loss, photon decay, coupling phase, and noise. In the weak-coupling regime, the full delay-rate description is reduced to an effective timing-phase model, yielding an explicit repetition-rate locking relation together with a delay-dependent locking condition. The resulting framework predicts the collapse of initially detuned pulse trains into a shared repetition frequency, controlled tuning of the locked frequency through the external delay, suppression of relative timing jitter in the synchronized state, and the emergence of mutual optical coherence only when additional carrier-phase requirements are satisfied. This hierarchy clearly separates repetition-rate synchronization from full optical phase coherence and is consistent with the principal qualitative trends reported in recent QD-PMLL synchronization experiments. The theory therefore provides a compact but physically transparent design framework for analyzing and engineering future chip-scale coupled-comb systems.

## References

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